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# Korovkin-type theorems in $C^*$ -algebras.

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## abstract

Korovkin の定理:

$C_r[a, b]$  の正值線型写像の列  $\{\Phi_n\}$  が

$\Phi_n x^j \rightarrow x^j \quad (n \rightarrow \infty) \quad j = 0, 1, 2,$  ならば  $\Phi_n u \rightarrow u \quad (u \in C_r[a, b])$ .

更に Korovkin は上の定理は  $\{1, x, x^2\}$  の代わりに Tschebyshev system でも成立する事を示した。このような集合は Korovkin 集合と呼ばれるが、これについては多くの研究結果がある。一般 choquet 境界を用いて  $C^*$ -algebra の場合にも拡張された。

Priestley(1976) は  $C^*$ -algebra  $\mathcal{A} \ni 1$  における正值写像  $\Phi_n(1) \leq 1$  について次の結果を得た。 $\{, \}$  は Jordan product である。

$\{a \in \mathcal{A} : \Phi_n(a) \rightarrow a, \Phi_n(a^2) \rightarrow a^2, \Phi_n(\{a^*, a\}) \rightarrow \{a^*, a\}\}$  は  $J^*$ -subalgebra.

定義。  $C^*$ -algebra  $\mathcal{A} \ni 1$  上の線型写像  $\Phi$  が  $\Phi(a)^* \Phi(a) \leq \Phi(a^* a) \quad (a \in \mathcal{A})$  をみたしているとき Schwarz map といわれる。

Robertson は Priestley の研究を次のように発展させた。

$\{\Phi_n\}$  を Schwarz map の列で  $\Phi_n(1) \leq 1$  とする。

$K := \{a \in \mathcal{A} : \Phi_n(x) \rightarrow x \quad (n \rightarrow \infty) \quad (x = a, a^* a, a a^*)\}$  は  $C^*$ -algebra

これから  $C(X) \supseteq M$  が  $X$  の点を分離すれば  $M \cup \{|h|^2 : h \in M\}$  は Korovkin 集合になる。

Limaye, Namboodiri は次のように拡張した。

$D := \{a \in \mathcal{A} : \Phi_n(a) \rightarrow a, \Phi_n(a^* a) \rightarrow a^* a\}$  は norm closed subalgebra である。

彼らは正值写像は、Jordan product  $\{, \}$  について Schwarz 写像になる事から 正值写像  $\Phi_n(1) \leq 1$  について次の結果を得た。

$C := \{a \in \mathcal{A} : \Phi_n(a) \rightarrow a, \Phi_n(\{a^*, a\}) \rightarrow \{a^*, a\}\}$  は  $J^*$ -subalgebra.

これらの定理は 可環である場合に知られている Korovkin 集合 についての結果をほとんど含んでいない。この講演 (論文) の目標は非可環のばあいこれを

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実現することである。そのために上記の定理の証明を明確にすることから始める。

### 1. Introduction

The Korovkin theorem [6] says that an arbitrary sequence  $\{\Phi_n\}$  of positive linear maps on  $C_r([a, b])$ , the Banach algebra of continuous real valued functions on  $[a, b]$ , strongly converges to the identity map if  $\Phi_n u \rightarrow u (n \rightarrow \infty)$  for  $u(t) = 1, t, t^2$ . Moreover Korovkin showed that this result holds for any Tshebyshev system  $\{f_0, f_1, f_2\}$  of order 2 instead of  $\{1, t, t^2\}$ . Here  $\{f_0, f_1, f_2\}$  is called Tshebyshev system of order 2 if  $a_0 f_0(x) + a_1 f_1(x) + a_2 f_2(x) = 0$  has at most 2 zeros in this interval. A subset  $K$  of  $C_r([a, b])$  is called a *Korovkin set*, provided an arbitrary sequence  $\{\Phi_n\}$  of positive linear maps on  $C_r([a, b])$  strongly converges to the identity map if  $\Phi_n u \rightarrow u (n \rightarrow \infty)$  for every  $u \in K$ . A lot of Korovkin sets are known. (H. Watanabe, T. Nishishiraho)

DEFINITION Let  $1 \in M \subseteq C(X)$  and set  $S(M) = \{l \in M^* : l(1) = 1 = ||l||\}$ . It is clear  $\hat{x} \in S(M)$  for every  $x \in X$ . The *Choquet boundary*  $B_M$  is  $\{x : \hat{x} \text{ is a extremepoint of } S(M)\}$ .

Wulbert has shown that if  $1 \in M \subseteq C(X)$  and if  $B_M = X$ , then  $M$  is a Korovkin set.

DEFINITION For a normed space  $E$  and for its subspace  $M$ , the *generalized Choquet boundary* is

$$B_M = \{l \in \text{ext}S(E^*) : l|_M \in \text{ext}S(M^*)\}.$$

Recently Operators of the property like  $I$  are investigated by S.Takahashi, Izuchi, Takagi, S.Watanabe.

The Korovkin theorem was extended to non-commutative  $C^*$ -algebras. (H. Choda- M.Echigo, S.Takahashi, J. Fujii)

Let  $\mathcal{A}$  be a  $C^*$ -algebra with an identity 1. A positive linear map  $\Phi$  on  $\mathcal{A}$  is called a *Schwarz map* if it satisfies  $\Phi(a)^* \Phi(a) \leq \Phi(a^*a)$  for every  $a \in \mathcal{A}$ . It is well-known that if  $\mathcal{A}$  is commutative then every contractive positive linear map is a Schwarz map. Robertson [11] has proved that, for a sequence  $\{\Phi_n\}$  of Schwarz maps, the set  $\{a \in \mathcal{A} : \Phi_n(x) \rightarrow x (n \rightarrow \infty) \text{ for } x = a, a^*a, aa^*\}$  is a  $C^*$ -subalgebra. As a corollary he also stated that for a sequence  $\{\Phi_n\}$  of contractive positive linear maps on the commutative  $C^*$ -algebra  $C(X)$  of continuous complex valued functions on a compact Hausdorff space  $X$ , the set  $\{u \in C(X) : \Phi_n(u) \rightarrow u, \Phi_n(|u|^2) \rightarrow |u|^2\}$  is a  $C^*$ -subalgebra. By identifying  $C_r(X)$  with the subalgebra of  $C(X)$ , the Stone-Weierstrass theorem shows that this contains the Korovkin theorem.

Let us recall that if  $\mathcal{B}$  is a  $C^*$ -subalgebra of  $C(X)$  and if for any point  $x \in X$  there is a  $f \in \mathcal{B}$  such that  $f(x) \neq 0$  and if  $\mathcal{B}$  separates  $X$ , then  $\mathcal{B} = C(X)$ .

Limaye and Namboodiri[7] have shown that for a sequence  $\{\Phi_n\}$  of Schwarz maps and a  $*$ -homomorphism  $\Phi$ , the set  $\{a \in \mathcal{A} : \Phi_n(a) \rightarrow \Phi(a), \Phi_n(a^*a) \rightarrow \Phi(a^*a)\}$

$\Phi(a^*a)$  is a closed (not necessarily  $*$ -closed) subalgebra and that  $\{a \in \mathcal{A} : \Phi_n(x) \rightarrow \Phi(x) \text{ for } x = a, a^*a, aa^*\}$ , the intersection of this subalgebra and its adjoint, is a  $C^*$ -subalgebra. By the Kadison theorem a contractive positive linear map  $\Phi$  satisfies  $\{\Phi(a)^*, \Phi(a)\} \leq \Phi(\{a^*, a\})$  for all  $a \in \mathcal{A}$ , where  $\{, \}$  is the Jordan product, i.e.,  $\{x, y\} = xy + yx$ .

Limaye and Namboodiri [8] have shown that, for a sequence  $\{\Phi_n\}$  of positive linear maps and a  $*$ -homomorphism  $\Phi$ , the set  $\{a \in \mathcal{A} : \Phi_n(a) \rightarrow \Phi(a), \Phi_n(\{a^*, a\}) \rightarrow \Phi(\{a^*, a\})\}$  is a  $*$ -closed, norm closed subspace which is also closed with respect to the Jordan product.

A continuous real valued function  $f(t)$  on  $[0, \infty)$  is called an *operator monotone function* if  $f(a) \geq f(b)$  whenever  $a \geq b \geq 0$ ,  $a, b \in \mathcal{A}$ . This function is characterized as follows:  $f$  is an operator function on  $[0, \infty)$  if and only if  $f$  has an analytic extension  $f(z)$  to the upper half plane such that  $\text{Im } f(z) > 0$  for  $\text{Im } z > 0$ . Therefore if  $f$  is an operator function, then so are  $f(\sqrt{t})^2$  and  $f(1/t)^{-1}$ .  $t^p$  ( $0 < p \leq 1$ ) and  $\log(t+1)$  are operator monotone functions. It is well-known that an operator monotone function is increasing and concave.

The aim of this paper is to give estimates of the norms related to schwarz maps and to extend Korovkin-type theorems by using operator monotone functions. These estimates seem to be very useful for studying Korovkin-type theorems in a non-commutative  $C^*$ -algebra; for instance we will give a quite simple proofs for many results given above.

## 2. generalized Schwarz maps.

Let  $\mathcal{A}$  be a  $C^*$ -algebra with a unit 1. A linear map  $\Phi$  is called a *Schwarz map* if  $\Phi(a)^*\Phi(a) \leq \Phi(a^*a)$  for every  $a \in \mathcal{A}$ , and a positive linear map  $\Psi$  with  $\Psi(1) \leq 1$  was called a *Jordan-Schwarz map* in [3], since it satisfies  $\{\Psi(a)^*, \Psi(a)\} \leq \Psi(\{a^*, a\})$  as we mentioned in the previous section. To investigate two cases given above all at once and to extend them, we consider the following binary operation  $\circ$  in  $\mathcal{A}$ :

$$\left\{ \begin{array}{l} (\alpha x + \beta y) \circ z = \alpha x \circ z + \beta y \circ z \quad (\alpha, \beta \in \mathbb{C}, x, y, z \in \mathcal{A}) ; \\ (x \circ y) \circ z = x \circ (y \circ z) ; \\ (x \circ y)^* = y^* \circ x^* ; \\ x^* \circ x \geq 0 ; \\ \text{there is a real number } M \text{ such that } \|x \circ y\| \leq M\|x\| \|y\|. \end{array} \right.$$

One may regard this binary operation as the ordinary product or the Jordan product.

Beckhoff [3] called a  $*$ -closed and norm-closed subspace of  $\mathcal{A}$  which is also closed with respect to the Jordan product a  *$J^*$ -subalgebra* of  $\mathcal{A}$ .

We call a linear subspace  $\mathcal{B} \subseteq \mathcal{A}$  a  *$\circ$ -subalgebra* if  $x \circ y \in \mathcal{B}$ , whenever

$x, y \in \mathcal{B}$ , and  $\circ^*$ -subalgebra if  $\mathcal{B}$  is a  $\circ$ -subalgebra and  $*$ -closed.

If a  $\circ^*$ -subalgebra is complete, that is norm-closed, then it is called a *complete  $\circ^*$ -subalgebra*.

**Definition.** A linear map  $\Phi : \mathcal{A} \rightarrow \mathcal{A}$  is called a *generalized Schwarz map* w.r.t.  $\circ$  if  $\Phi$  satisfies

$$\Phi(x^*) = \Phi(x)^* \text{ and } \Phi(x^*) \circ \Phi(x) \leq \Phi(x^* \circ x) \text{ for every } x \in \mathcal{A}.$$

We remark that a generalized Schwarz map  $\Phi$  is not necessarily positive (that  $\Phi$  is positive means  $\Phi(a) \geq 0$  whenever  $a \geq 0$ ).

**Definition.** A generalized Schwarz map  $\Phi$  w.r.t.  $\circ$  is called a  *$*$ -homomorphism* w.r.t.  $\circ$  if  $\Phi(x)^* \circ \Phi(x) = \Phi(x^* \circ x)$  for every  $x \in \mathcal{A}$ .

Let us note that if  $\Phi$  is a  $*$ -homomorphism w.r.t.  $\circ$ , then by a polarization

$$4x^* \circ y = \sum_{n=0}^3 i^n (i^n x + y)^* \circ (i^n x + y),$$

we deduce  $\Phi(x) \circ \Phi(y) = \Phi(x \circ y)$  for every  $x, y \in \mathcal{A}$ . It is clear that if  $\circ$  is the original product in  $\mathcal{A}$ , then a  $*$ -homomorphism w.r.t.  $\circ$  is a  $*$ -homomorphism in the ordinal sense, and that if  $\circ$  is the Jordan product, then a  $*$ -homomorphism w.r.t.  $\circ$  is a  $C^*$ -homomorphism in the ordinal sense. A bounded linear functional  $\phi$  of  $\mathcal{A}$  is called a *state* if  $\phi$  is positive and  $\phi(1) = 1$ .

**Theorem 2.1.** Let  $\Phi$  be a generalized Schwarz map w.r.t.  $\circ$  on  $\mathcal{A}$ . For  $x, y \in \mathcal{A}$  set

$$X := \Phi(x^* \circ x) - \Phi(x)^* \circ \Phi(x) \geq 0,$$

$$Y := \Phi(y^* \circ y) - \Phi(y)^* \circ \Phi(y) \geq 0,$$

$$Z := \Phi(x^* \circ y) - \Phi(x)^* \circ \Phi(y).$$

Then we have

$$|\phi(Z)| \leq \phi(X)^{\frac{1}{2}} \phi(Y)^{\frac{1}{2}} \quad (1)$$

for every state  $\phi \in \mathcal{A}'$ . Further we have

$$\frac{1}{2} \|Z\| \leq \|X\|^{\frac{1}{2}} \|Y\|^{\frac{1}{2}} \quad (2)$$

*Proof.* For every complex number  $\alpha$ , we have  
 $0 \leq \Phi((x + \alpha y)^* \circ (x + \alpha y)) - \Phi(x + \alpha y)^* \circ \Phi(x + \alpha y) = X + \alpha Z + \overline{\alpha} \overline{Z} + |\alpha|^2 Y$ ,  
 from which it follows that  
 $0 \leq \phi(X) + 2\operatorname{Re} \alpha \phi(Z) + |\alpha|^2 \phi(Y)$  for every state  $\phi \in \mathcal{A}'$ . Thus we can easily

get (1). Since  $\sup\{\phi(Z) : \phi \text{ is a state of } \mathcal{A}\}$  is the numerical radius  $w(Z)$ , from (1) we obtain  $w(Z) \leq w(X)^{\frac{1}{2}}w(Y)^{\frac{1}{2}}$ .

It is well-known that  $\frac{1}{2}\|a\| \leq w(a) \leq \|a\|$  for every  $a \in \mathcal{A}$ .

Thus we obtain (2).  $\square$

From the inequality (2) we can easily prove results mentioned in the first section.

**Proposition 2.2.** *Let  $\{\Phi_n\}$  be a sequence of generalized Schwarz maps w.r.t.  $\circ$  on  $\mathcal{A}$  with  $\|\Phi_n\| \leq 1$ , and  $\Phi$  a  $*$ -homomorphism w.r.t.  $\circ$  on  $\mathcal{A}$  with  $\|\Phi\| \leq 1$ . Then the set  $D := \{x \in \mathcal{A} : \|\Phi_n(x) - \Phi(x)\| \rightarrow 0, \|\Phi_n(x^* \circ x) - \Phi(x^* \circ x)\| \rightarrow 0 \text{ as } n \rightarrow \infty\}$  is a complete  $\circ$ -subalgebra.*

*Proof.* Suppose  $x \in D$ . From the definition of  $\circ$ , it follows that  
 $0 \leq \|\Phi_n(x)^* \circ \Phi_n(x) - \Phi(x)^* \circ \Phi(x)\|$   
 $\leq M\|\Phi_n(x)^* - \Phi(x)^*\| \|\Phi(x)\| + M\|\Phi_n(x) - \Phi(x)\| \rightarrow 0.$   
 This and

$$\Phi_n(x^* \circ x) \rightarrow \Phi(x^* \circ x) = \Phi(x)^* \circ \Phi(x)$$

imply

$$\|\Phi_n(x^* \circ x) - \Phi_n(x)^* \circ \Phi_n(x)\| \rightarrow 0 \quad (n \rightarrow \infty).$$

Thus for every  $y \in \mathcal{A}$ , in virtue of (2) we get

$$\|\Phi_n(x^* \circ y) - \Phi_n(x)^* \circ \Phi_n(y)\| \rightarrow 0 \quad (n \rightarrow \infty),$$

which implies that

$$\Phi_n(x^* \circ y) \rightarrow \Phi(x)^* \circ \Phi(y) \text{ if } x \in D \text{ and } \Phi_n(y) \rightarrow \Phi(y).$$

From this one can see that  $x \circ y \in D$  if  $x, y \in D$ . Since  $\{\Phi_n\}$  is uniformly bounded,  $D$  is complete.  $\square$

**Corollary 2.3.** Under the above condition the set  $D \cap D^*$  is a complete  $\circ^*$ -subalgebra.

**Remark.** Since every bounded linear functional on  $\mathcal{A}$  is a linear combination of at most four states of  $\mathcal{A}$ , a sequence  $\{a_n\}$  of  $\mathcal{A}$  weakly converges to  $a$  if and only if  $\phi(a_n) \rightarrow \phi(a)$  for every state  $\phi$ . By using (1) we can see that  
 $D_1 := \{x \in \mathcal{A} : \Phi_n(x) \rightarrow \Phi(x) \text{ (} w \text{)}, \Phi_n(x^* \circ x) \rightarrow \Phi(x^* \circ x) \text{ (} w \text{)}\}$

is a complete  $\circ$ -subalgebra, and hence that  $D_1 \cap D_1^*$  is a complete  $\circ^*$ -subalgebra.

Proposition 2.2, Corollary 2.3 and Remark were proved in [7] [8] [11] when  $\circ$  is the original product or the Jordan product in  $\mathcal{A}$ , but the above proof seems to be simple.

We denote the  $\circ^*$ -subalgebra of  $\mathcal{A}$  generated by a subset  $S$  of  $\mathcal{A}$  by  $\mathcal{J}^*(S, \circ)$  or simply by  $\mathcal{J}^*(S)$ . We define the *Korovkin closure*  $Kor_{\mathcal{A}}(S)$  of a subset  $S \subseteq \mathcal{A}$  as follows :  $Kor_{\mathcal{A}}(S)$  is the set of all  $x \in \mathcal{A}$  such that for every sequence  $\{\Phi_n\}$  of *positive* generalized Schwarz maps w.r.t.  $\circ$  with  $\|\Phi_n\| \leq 1$ ,  $\Phi_n x \rightarrow x$  whenever  $\Phi_n a \rightarrow a$  for every  $a \in S$ . Here the convergence means convergence in the norm topology. From this definition the next follows :

**Lemma 2.4.**  $Kor_{\mathcal{A}}(S) \subseteq Kor_{\mathcal{A}}(T)$  if  $S \subseteq T$ .  $Kor_{\mathcal{A}}(S) \subseteq Kor_{\mathcal{A}}(T)$  if  $S \subseteq Kor_{\mathcal{A}}(T)$ .

**Corollary 2.5.** For a subset  $S \subseteq \mathcal{A}$ , we have

$$\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_1), \text{ where } S_1 := S \cup \{x^* \circ x : x \in S\} \cup \{x \circ x^* : x \in S\}. \quad (3)$$

*Proof.* Fix a sequence  $\{\Phi_n\}$  of positive generalized Schwarz maps w.r.t.  $\circ$  with  $\|\Phi_n\| \leq 1$  such that  $\Phi_n(t) \rightarrow t$  for every  $t \in S_1$ . We have only to show  $\Phi_n(t) \rightarrow t$  for every  $t \in \mathcal{J}^*(S)$ . By Corollary 2.3, the set  $\{x \in \mathcal{A} : \Phi_n(x) \rightarrow x, \Phi_n(x^* \circ x) \rightarrow x^* \circ x, \Phi_n(x \circ x^*) \rightarrow x \circ x^*\}$  is a  $\circ^*$ -subalgebra. Since it contains  $S$ , it contains  $\mathcal{J}^*(S)$  too. Thus we have  $\Phi_n(t) \rightarrow t$  for every  $t \in \mathcal{J}^*(S)$ .  $\square$

**Theorem 2.6.** Let  $f$  be an operator monotone function on  $[0, \infty)$  with  $f(0) = 0$  and  $\lim_{t \rightarrow \infty} f(t) = \infty$ . Set  $g = f^{-1}$ . Then for a subset  $S$  of  $\mathcal{A}$  we have

$$\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_2), \text{ where } S_2 := S \cup \{g(x^* \circ x) : x \in S\} \cup \{g(x \circ x^*) : x \in S\}. \quad (4)$$

*Proof.* Let  $\{\Phi_n\}$  be a sequence of positive generalized Schwarz maps w.r.t.  $\circ$  with  $\|\Phi_n\| \leq 1$  such that  $\Phi_n(t) \rightarrow t$  for every  $t \in S_2$ . It was shown in [4] [5] that

$$\Phi_n(f(a)) \leq f(\Phi_n(a)) \text{ for every } a \geq 0, \quad (5)$$

which implies

$$0 \leq \Phi_n(x^* \circ x) - \Phi_n(x)^* \circ \Phi_n(x) \leq f(\Phi_n(g(x^* \circ x))) - \Phi_n(x)^* \circ \Phi_n(x)$$

for every  $x$ . From  $\Phi_n(g(x^* \circ x)) \rightarrow g(x^* \circ x)$ , it follows that  $f(\Phi_n(g(x^* \circ x))) \rightarrow x^* \circ x$ . Thus the right side of the above inequality converges to 0, from which it follows that

$$\lim \Phi_n(x^* \circ x) = \lim \Phi_n(x)^* \circ \Phi_n(x) = x^* \circ x^*.$$

Similarly we can get  $\lim \Phi_n(x \circ x^*) = x \circ x^*$ .

Thus we have shown that  $\Phi_n(t) \rightarrow t$  for every  $t$  in  $S_1$  which was given in Corollary 2.5, that is, we have shown  $S_1 \subseteq \text{Kor}_{\mathcal{A}}(S_2)$ . By (3) and Lemma 2.4 we have  $J^*(S) \subseteq \text{Kor}_{\mathcal{A}}(S_1) \subseteq \text{Kor}_{\mathcal{A}}(S_2)$ . Consequently we get (4).  $\square$

**Theorem 2.7.** *Let  $g$  be a function given in Theorem 2.6. For a finite set  $S = \{s_1, \dots, s_n\}$ , we have*

$$J^*(S) \subseteq \text{Kor}_{\mathcal{A}}(S_3), \text{ where } S_3 = S \cup \left\{ g\left(\sum_{i=1}^n (s_i^* \circ s_i + s_i \circ s_i^*)\right) \right\}. \quad (6)$$

*Proof.* Let us take an arbitrary sequence  $\Phi_n$  of positive generalized Schwarz maps w.r.t  $\circ$  with  $\|\Phi_n\| \leq 1$  such that  $\{\Phi_n(t)\} \rightarrow t$  for every  $t \in S_3$ . For each  $i$

$$\begin{aligned} 0 &\leq \Phi_n(s_i^* \circ s_i) - \Phi_n(s_i)^* \circ \Phi_n(s_i) \leq \sum_{j=1}^n \{ \Phi_n(s_j^* \circ s_j) - \Phi_n(s_j)^* \circ \Phi_n(s_j) \} \\ &\leq \Phi_n\left(\sum_j (s_j^* \circ s_j + s_j \circ s_j^*)\right) - \sum_j \{ \Phi_n(s_j)^* \circ \Phi_n(s_j) + \Phi_n(s_j) \circ \Phi_n(s_j)^* \} \\ &\leq f\left(\Phi_n\left(g\left(\sum_j (s_j^* \circ s_j + s_j \circ s_j^*)\right)\right)\right) - \sum_j \{ \Phi_n(s_j)^* \circ \Phi_n(s_j) + \Phi_n(s_j) \circ \Phi_n(s_j)^* \}. \end{aligned}$$

Since the right side converges to 0,  $\Phi_n(s_i^* \circ s_i)$  converges to  $s_i^* \circ s_i$ . Similarly we can see that  $\Phi_n(s_i \circ s_i^*)$  converges to  $s_i \circ s_i^*$ . Thus we have shown that  $S_1 := S \cup \{s^* \circ s : s \in S\} \cup \{s \circ s^* : s \in S\} \subseteq \text{Kor}_{\mathcal{A}}(S_3)$ .

By (3) and Lemma 2.4, we get (6).  $\square$

**Theorem 2.8.** *Under the same assumption as Theorem 2.6, we have*

$$J^*(S) \subseteq \text{Kor}_{\mathcal{A}}(S \cup \{g(x^* \circ x + x \circ x^*) : x \in S\}). \quad (7)$$

*Proof.* By substituting  $x$  for  $s_i$  in the inequalities of the proof of Theorem 2.7, we get

$$\begin{aligned} 0 &\leq \Phi_n(x^* \circ x) - \Phi_n(x)^* \circ \Phi_n(x) \\ &\leq f(\Phi_n(g(x^* \circ x + x \circ x^*))) - \{ \Phi_n(x)^* \circ \Phi_n(x) + \Phi_n(x) \circ \Phi_n(x)^* \}. \end{aligned}$$



Thus in the same fashion as Theorem 2.6 we can get (7).  $\square$

In the above three theorems we needed conditions  $f(0) = 0$ ,  $f(\infty) = \infty$  in order that  $f^{-1} = g$  is defined on  $[0, \infty)$  and that (5) is valid for every positive map. However, when we consider the case of  $1 \in S$ , we can loose the condition  $f(0) = 0$ .

**Theorem 2.9.** *Suppose  $1 \in S \subseteq \mathcal{A}$ . Let  $f$  be an operator monotone function defined on  $[0, \infty)$  such that  $f(0) \leq 0$ ,  $f(\infty) = \infty$ . Set  $g = f^{-1}$ . Then we have  $\mathcal{J}^*(S) \subseteq \text{Kor}_{\mathcal{A}}(S_2)$ , where  $S_2 = S \cup \{g(x^* \circ x) | x \in S\} \cup \{g(x \circ x^*) | x \in S\}$ .*

*Proof.* Let us take an arbitrary sequence  $\{\Phi_n\}$  of positive generalized Schwarz maps w.r.t  $\circ$  with  $\|\Phi_n\| \leq 1$  such that  $\Phi_n(t) \rightarrow t$  for every  $t \in S_2$ . By (5) we get

$$\Phi_n(f(a) - f(0)1) \leq f(\Phi_n(a)) - f(0)1 \text{ for every } a \geq 0,$$

and hence

$$\Phi_n(a) = \Phi_n(f(g(a))) \leq f(\Phi_n(g(a))) - f(0)(1 - \Phi_n(1)).$$

From this, for every  $x \in S$  we deduce

$$\begin{aligned} 0 &\leq \Phi_n(x^* \circ x) - \Phi_n(x)^* \circ \Phi_n(x) \\ &\leq f(\Phi_n(g(x^* \circ x))) - f(0)(1 - \Phi_n(1)) - \Phi_n(x)^* \circ \Phi_n(x). \end{aligned}$$

Since the bigger side in the above converges to 0, we obtain that  $\Phi_n(x^* \circ x) \rightarrow x^* \circ x$ . Similarly we can get  $\Phi_n(x \circ x^*) \rightarrow x \circ x^*$ . By (3) we get  $\mathcal{J}^*(S) \subseteq \text{Kor}_{\mathcal{A}}(S_2)$ .  $\square$

In the same fashion as the above proof, we can easily extend Theorem 2.7 and Corollary 2.8 to the case of  $1 \in S$  as follows :

**Theorem 2.10.** *Let  $S = \{s_1, \dots, s_n\}$  be a subset of  $\mathcal{A}$  and include 1. Let  $f$  be an operator monotone function defined on  $[0, \infty)$  such that  $f(0) \leq 0$ ,  $f(\infty) = \infty$ . Set  $g = f^{-1}$ . Then we have  $\mathcal{J}^*(S) \subseteq \text{Kor}_{\mathcal{A}}(S_3)$ , where  $S_3 = S \cup \{g(\sum_{i=1}^n (s_i^* \circ s_i + s_i \circ s_i^*))\}$ .*

**Corollary 2.11.** *Under the same assumption as Theorem 2.9, we have  $\mathcal{J}^*(S) \subseteq \text{Kor}_{\mathcal{A}}(S \cup \{g(x^* \circ x + x \circ x^*) : x \in S\})$ .*

**Remark.** In the above theorems we studied not the universal Korovkin closures (the definition is given below) but the Korovkin closures, that is, the case where  $\Phi_n \rightarrow 1$  instead of  $\Phi_n \rightarrow \Phi$ . To get the same conclusions for  $\Phi$  as theorems, we would have to assume that  $\Phi$  is  $*$ -homomorphism w.r.t.  $\circ$  and  $*$ -homomorphism in the ordinary sense because of  $\Phi(g(a)) = g(\Phi(a))$ ; we thought it is a bit complicated assumption. If a binary operation  $\circ$  is the ordinary product or the Jordan product, then  $*$ -homomorphism in the ordinary sense is a  $*$ -homomorphism w.r.t.  $\circ$  too. Now we consider this case. Let us define the *universal Korovkin closure*  $Kor_{\mathcal{A}}^u(S)$  of a subset  $S \subseteq \mathcal{A}$  as follows :  $Kor_{\mathcal{A}}^u(S)$  is the set of all  $x \in \mathcal{A}$  such that for every  $*$ -homomorphism  $\Phi$  and for every sequence  $\{\Phi_n\}$  of *positive* generalized Schwarz maps w.r.t.  $\circ$  with  $\|\Phi_n\| \leq 1$ ,  $\Phi_n x \rightarrow \Phi x$  whenever  $\Phi_n(a) \rightarrow \Phi(a)$  for every  $a \in S$ . When  $\circ$  is the ordinary product or the Jordan product, it is not difficult to see that we can substitute  $Kor_{\mathcal{A}}^u(S)$  for  $Kor_{\mathcal{A}}$  in the above theorems.

At the end of this section we consider the case where  $\circ$  is the ordinary product, and we extend the Robertson's theorem in a visible form :

**Theorem 2.12.** *Let  $\{\Phi_n\}$  be a sequence of Schwarz maps and  $\Phi$  a  $*$ -homomorphism, and let  $f$  be an operator monotone function on  $[0, \infty)$  with  $f(0) = 0$ ,  $f(\infty) = \infty$ . Set  $g = f^{-1}$ . Then the set  $C := \{a \in \mathcal{A} : \Phi_n(x) \rightarrow \Phi(x) \text{ for } x = a, g(a^*a), g(aa^*)\}$  is a  $C^*$ -subalgebra.*

*Proof.* That  $\Phi_n(a)$  converges to  $\Phi(a)$  implies  $\Phi_n(a)^* \Phi_n(a) \rightarrow \Phi(a)^* \Phi(a)$ , and that  $\Phi_n(g(a^*a))$  converges to  $\Phi(g(a^*a))$  implies

$$f(\Phi_n(g(a^*a))) \rightarrow f(\Phi(g(a^*a))) = \Phi(a^*a) = \Phi(a)^* \Phi(a).$$

Thus we have  $f(\Phi_n(g(a^*a))) - \Phi_n(a)^* \Phi_n(a) \rightarrow 0$ . From (5) it follows that

$$0 \leq \Phi_n(a^*a) - \Phi_n(a)^* \Phi_n(a) \leq f(\Phi_n(g(a^*a))) - \Phi_n(a)^* \Phi_n(a).$$

Hence we get  $\Phi_n(a^*a) \rightarrow \Phi(a^*a)$ . Similary we can get  $\Phi_n(aa^*) \rightarrow \Phi(aa^*)$ .

Thus  $C \subseteq D \cap D^*$ , where  $D$  is given in Proposition 2.2. Conversely, since  $D \cap D^*$  is a  $C^*$ -subalgebra (Corollary 2.3),  $D \cap D^* \subseteq C$ . Consequently  $C$  is a  $C^*$ -subalgebra.  $\square$

### 3. Korovkin sets in $C(X)$ .

Let  $X$  be a compact Hausdorff space and  $C(X)$  a  $C^*$ -algebra of all complex valued continuous functions. Though we treat only complex algebras, the results which will be gotten for complex algebras in this section hold for real algebras too. Since a positive linear map  $\Phi$  on  $C(X)$  satisfies  $|\Phi(fg)|^2 \leq \Phi(|f|^2)\Phi(|g|^2)$ ,  $\Phi$  is a Schwarz map with respect to the ordinary product if  $\Phi(1) \leq 1$ . A subset  $S$  of  $C(X)$  is called a *Korovkin set* if  $K_{C(X)}(S) = C(X)$ .

Here  $K_{C(X)}$  is the set of every  $x \in C(X)$  which satisfies that  $\Phi_n(x) \rightarrow x$  for every sequence of Schwarz maps (i.e,  $0 \leq \Phi_n, \Phi_n(1) \leq 1$ ) such that  $\Phi_n(s) \rightarrow s$  for all  $s \in S$ .  $C^*(S)$  stands for the  $C^*$ -algebra generated by  $S$ .

**Theorem 3.1.** *Let  $f$  be an operator monotone function defined on  $[0, \infty)$  such that  $f(0) \leq 0$ ,  $f(\infty) = \infty$ , and set  $g = f^{-1}$ . Then for a subset  $S$  of  $C(X)$*

$$C^*(S) \subseteq K_{C(X)}(S \cup \{g(|u|^2) : u \in S\}) \text{ if } f(0) = 0, \text{ or } 1 \in S.$$

*Proof.* This follows from Theorems 2.6, 2.9.  $\square$

**Theorem 3.2.** *Let  $f$  be an operator monotone function defined on  $[0, \infty)$  with  $f(0) \leq 0$ ,  $f(\infty) = \infty$ , and set  $g = f^{-1}$ . If a finite subset  $S = \{u_1, \dots, u_m\} \subseteq C(X)$  separates strongly the points of  $X$ , then  $S \cup \{g(|u_1|^2 + \dots + |u_m|^2)\}$  is a Korovkin set if  $f(0) = 0$ , or  $1 \in S$ .*

*Proof.* By Theorems 2.7, 2.10, we have  $C^*(S) \subseteq K_{C(X)}(S \cup \{g(|u_1|^2 + \dots + |u_m|^2)\})$  if  $f(0) = 0$  or  $1 \in S$ . From the Stone-Weierstrass theorem  $C^*(S) = C(X)$  follows.  $\square$

In [9], the above theorem was shown in the case where  $g(t) = t$ . The forms of Korovkin sets given above include many Korovkin sets in Appendix C of [1].

#### 4. Minimal norm ideals.

In this section we treat the minimal norm ideals of  $C^*$ -algebra  $\mathcal{B}(H)$  of all bounded operators on a Hilbert space  $H$ . Korovkin-type theory in the minimal norm ideals was studied in [10], [3]. Let  $\mathcal{I}$  be a minimal norm ideal with a symmetric norm  $\|\cdot\|$ , and  $\mathcal{A}$  a  $C^*$ -algebra generated by every compact operator and 1. We use the notation introduced in the second section. But we assume that a binary operation  $\circ$  defined on  $\mathcal{A}$  satisfies

$$\|x \circ y\| \leq M \|x\| \|y\|$$

instead of  $\|x \circ y\| \leq M \|x\| \|y\|$ .

**Theorem 4.1.** *Let  $\Phi$  be a generalized Schwarz map w.r.t.  $\circ$  on  $\mathcal{A}$ . For  $x, y \in \mathcal{A}$  set  $X := \Phi(x^* \circ x) - \Phi(x)^* \circ \Phi(x) \geq 0$ ,  $Y := \Phi(y^* \circ y) - \Phi(y)^* \circ \Phi(y) \geq 0$ ,  $Z := \Phi(x^* \circ y) - \Phi(x)^* \circ \Phi(y)$ .*

*Then we have*

$$\frac{1}{2} \|Z\| \leq \|X\|^{\frac{1}{2}} \|Y\|^{\frac{1}{2}} \quad (8)$$

*Proof.* (1) implies  $|(Zu, u)| \leq (Xu, u)^{\frac{1}{2}}(Yu, u)^{\frac{1}{2}}$  for every  $u \in \mathcal{H}$ . By using the polarization :

$$4(Zu, v) = (Z(u+v), u+v) - (Z(u-v), u-v) + i(Z(u+iv), u+iv) - i(Z(u-iv), u-iv),$$

we get,

$$4|(Zu, v)| \leq (X(u+v), u+v)^{\frac{1}{2}}(Y(u+v), u+v)^{\frac{1}{2}} + (X(u-v), u-v)^{\frac{1}{2}}(Y(u-v), u-v)^{\frac{1}{2}} \\ + (X(u+iv), u+iv)^{\frac{1}{2}}(Y(u+iv), u+iv)^{\frac{1}{2}} + (X(u-iv), u-iv)^{\frac{1}{2}}(Y(u-iv), u-iv)^{\frac{1}{2}}$$

and hence in virtue of Schwarz inequality,

$$|(Zu, v)| \leq \{(Xu, u) + (Xv, v)\}^{\frac{1}{2}} \{(Yu, u) + (Yv, v)\}^{\frac{1}{2}}$$

for every  $u, v \in \mathcal{H}$ . Thus, for arbitrary orthonormal sets  $\{u_i\}, \{v_i\}$

$$|(Zu_i, v_i)| \leq \{(Xu_i, u_i) + (Xv_i, v_i)\}^{\frac{1}{2}} \{(Yu_i, u_i) + (Yv_i, v_i)\}^{\frac{1}{2}} \\ \leq \frac{1}{2}t\{(Xu_i, u_i) + (Xv_i, v_i)\} + \frac{1}{2}\frac{1}{t}\{(Yu_i, u_i) + (Yv_i, v_i)\}$$

for every  $t > 0$ , because of  $2\sqrt{\lambda\mu} = \min\{t\lambda + \frac{1}{t}\mu : t > 0\}$ . Now we consider this inequality to be an estimate of general terms of sequences. By taking the symmetric norm of these sequences, we get

$$\|(Zu_i, v_i)\| \leq \frac{1}{2}t\|(Xu_i, u_i) + (Xv_i, v_i)\| + \frac{1}{2}\frac{1}{t}\|(Yu_i, u_i) + (Yv_i, v_i)\| \leq t\|X\| + \frac{1}{t}\|Y\|.$$

Since

$$\sup\{\|(Zu_i, v_i)\| : \{u_i\}, \{v_i\}\} = \|Z\| \quad ([12]),$$

we have  $\|Z\| \leq (t\|X\| + \frac{1}{t}\|Y\|)$  for every  $t > 0$ , and hence  $\|Z\| \leq 2\|X\|^{\frac{1}{2}}\|Y\|^{\frac{1}{2}}$ .

□

**Theorem 4.2.** Let  $\mathcal{I}$  be a minimal ideal of  $\mathcal{B}(H)$  with the symmetric norm  $\|\cdot\|$ . Let  $\{\Phi_n\}$  be a sequence of generalized Schwarz maps w.r.t  $\circ$  on  $\mathcal{I}$  with  $\|\Phi_n\| \leq 1$ , and  $\Phi$  a  $*$ -homomorphism w.r.t  $\circ$  on  $\mathcal{I}$  with  $\|\Phi_n\| \leq 1$ . Then the set  $D = \{x \in \mathcal{I} : \|\Phi_n(x) - \Phi(x)\| \rightarrow 0, \|\Phi_n(x^* \circ x) - \Phi(x^* \circ x)\| \rightarrow 0\}$  is a  $\|\cdot\|$ -closed  $\circ$ -subalgebra of  $\mathcal{I}$ .

*Proof.* This theorem follows from (8) in the same way that Proposition 2.2 followed from (2). □

**Corollary 4.3.** Under the same condition as above,  $D \cap D^*$  is a  $\|\cdot\|$ -closed  $*$ -subalgebra of  $\mathcal{I}$ .

Theorem 4.2 was proved in [10] in the case where  $\mathcal{I}$  is the trace class, and other cases were described in [3]. But our proof is clearly simpler than the proof

of [10].

We want to extend the above. But it is not easy, because  $\|a_n - a\| \rightarrow 0$  does not necessarily imply  $\|a_n^{\frac{1}{p}} - a^{\frac{1}{p}}\| \rightarrow 0$ .

Therefore we could not get a theorem as Theorem 2.6. To get a slight extension we denote  $\| |A|^p \|_{\frac{1}{p}}$  by  $\|A\|_p$  for  $0 < p < \infty$ .

**Theorem 4.4.** *Let  $\mathcal{I}$  be a minimal norm ideal of  $\mathcal{B}(H)$  with the symmetric norm  $\|\cdot\|$ . Let  $\{\Phi_n\}$  be a sequence of generalized Schwarz maps w.r.t.  $\circ$  on  $\mathcal{I}$  with  $\|\Phi_n\| \leq 1$ . Let  $S$  be a subset of  $\mathcal{I}$  and  $\mathcal{I}_S^*$  a complete  $\circ^*$ -subalgebra generated by  $S$ . Then for an arbitrary integer  $m$*   
 $\mathcal{I}_S^* \subseteq \{x \in \mathcal{I} : \|\Phi_n x - x\| \rightarrow 0 \text{ whenever } \|\Phi_n(s) - s\| \rightarrow 0,$   
 $\|\Phi_n((s^* \circ s + s \circ s^*)^m) - (s^* \circ s + s \circ s^*)^m\|_{\frac{1}{m}} \rightarrow 0 \text{ for all } s \in S\}$  (9)

*Proof.* By the previous theorem we need only to show that if

$$\|\Phi_n((s^* \circ s + s \circ s^*)^m) - (s^* \circ s + s \circ s^*)^m\|_{\frac{1}{m}} \rightarrow 0 \ (n \rightarrow \infty),$$

then

$$\|\Phi_n(s^* \circ s) - s^* \circ s\| \rightarrow 0 \text{ and } \|\Phi_n(s \circ s^*) - s \circ s^*\| \rightarrow 0.$$

By the definition of the norm we have

$$\|\|\Phi_n((s^* \circ s + s \circ s^*)^m) - (s^* \circ s + s \circ s^*)^m\|_{\frac{1}{m}}\| \rightarrow 0.$$

By the Ando theorem [2] :

$$\|x^{\frac{1}{m}} - y^{\frac{1}{m}}\| \leq \|x - y\|^{\frac{1}{m}} \text{ for all } x, y \geq 0,$$

we obtain

$$\|\{\Phi_n((s^* \circ s + s \circ s^*)^m)\}^{\frac{1}{m}} - (s^* \circ s + s \circ s^*)\| \rightarrow 0.$$

Thus, from

$$\|\Phi_n(s^*) \circ \Phi_n(s) + \Phi_n(s) \circ \Phi_n(s)^* - (s^* \circ s + s \circ s^*)\| \rightarrow 0,$$

it follows that

$$\|\{\Phi_n(s^* \circ s + s \circ s^*)^m\}^{\frac{1}{m}} - \{\Phi_n(s^*) \circ \Phi_n(s) + \Phi_n(s) \circ \Phi_n(s)^*\}\| \rightarrow 0.$$

Since

$$\begin{aligned} 0 &\leq \{\Phi_n(s^* \circ s) - \Phi_n(s^*) \circ \Phi_n(s)\} + \{\Phi_n(s \circ s^*) - \Phi_n(s) \circ \Phi_n(s^*)\} \\ &= \Phi_n(s^* \circ s + s \circ s^*) - \{\Phi_n(s^*) \circ \Phi_n(s) + \Phi_n(s) \circ \Phi_n(s^*)\} \\ &\leq \{\Phi_n(s^* \circ s + s \circ s^*)^m\}^{\frac{1}{m}} - \{\Phi_n(s^*) \circ \Phi_n(s) + \Phi_n(s) \circ \Phi_n(s)^*\}, \end{aligned}$$

we deduce  $\|\{\Phi_n(s^* \circ s) - \Phi_n(s^*) \circ \Phi_n(s)\} \| \rightarrow 0$  and  $\|\{\Phi_n(s \circ s^*) - \Phi_n(s) \circ \Phi_n(s^*)\} \| \rightarrow 0$ . Here we used the fact that  $0 \leq a \leq b$  generally implies  $\|a\| \leq \|b\|$  : in fact  $0 \leq a \leq b$  implies that there is  $c \in \mathcal{B}(\mathcal{H})$  such that  $a = c^*bc$  and  $\|c\| \leq 1$ .  $\square$

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